Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(01)00074-0
J. Appl. Maths Mechs, Vol. 65, No. 4, pp. 697–702, 2001

0021-8928/01/$—see front matter

# THE OSCILLATIONS OF A PENDULUM IN A CIRCULAR ORBIT†

A. A. BUROV and S. Ya. STEPANOV

Moscow

e-mail: aburov@ccas.ru; stepsj@ccas.ru

(*Received* 17 *August* 2000)

The model problem of the motion in a central force field of a pendulum whose suspension point slides along a circular directrix with centre at an attracting centre is considered. The steady motions and conditions for their stability are determined. The results are used to investigate the motions of an orbital crane. 

To displace loads in space, with the aim of forming large-scale orbital configurations [1], one can use devices similar to a ground-based crane, where the load, like a pendulum, is suspended on a tether, while the suspension point moves along a directrix attached to an artificial satellite. We will assume that the directrix has the form of a circle with centre at an attracting centre and that the artificial satellite has a fixed orientation in an orbital system of coordinates. In this formulation one can obtain the time-optimal control, similar to that obtained in [2]. To construct it one must carry out a preliminary investigation of the oscillations of a pendulum when the suspension point moves with constant velocity. This paper investigates these oscillations in an accurate formulation.

The problem of the equilibria of a pendulum was considered previously in [3] in the case when the suspension point is fixed at the centre of mass of a satellite which moves in a circular orbit with a Kepler orbital angular velocity. The problem has been investigated in the satellite approximation [4] for an arbitrary arrangement of the suspension point in a satellite moving in a circular Kepler orbit (see also [5]).

## 1. FORMULATION OF THE PROBLEM AND THE EQUATIONS OF MOTION

Suppose a point $P$ moves uniformly in a circle of radius $R$ around an attracting centre $O$. A point $Q$ of mass $m$, moving in the plane of this circle, is suspended at a point $P$ on a weightless inextensible rod of length $l_*$. We will introduce a system of coordinates $Oxy$ connected to the point $P$, which rotates uniformly with angular velocity $\theta'$ about the point $O$ (here and henceforth the prime denotes a derivative with respect to time). In this system of coordinates the vectors $OP$ and $PQ$ have coordinates $(0, R)$ and $(l_* \sin\varphi, -l_* \cos\varphi)$ (Fig. 1).

Suppose

$$s = \sin\varphi,\quad c = \cos\varphi,\quad r_*^2 = R^2 + l_*^2 - 2Rl_*c$$

The components of the absolute velocity of the point $Q$ along the $x$ and $y$ axes have the form

$$V_x = l_*c\varphi' + l_*'s - (R - l_*c)\theta',\quad V_y = l_*s\varphi' - l_*'c + l_*s\theta'$$

The kinetic energy of the point mass $Q$, the force function and the Lagrange function have the form

$$T = \frac{1}{2}m(V_x^2 + V_y^2) = \frac{1}{2}m(l_*^2\varphi'^2 + l_*'^2 + l_*^2\theta'^2 + R^2\theta'^2) + \\ +m[l_*^2\varphi'\theta' - R(l_*c\varphi' + dl_*s + l_*c\theta')\varphi'] \qquad (1.1)$$

†*Prikl. Mat. Mekh.* Vol. 65, No. 4, pp. 714–719, 2001.

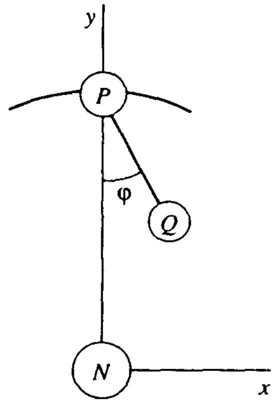


Fig. 1

$$U = -\frac{m\rho_*^3}{r_*}\theta'^2, \quad L_* = T + U = mR^2\theta'^2 L \tag{1.2}$$

$$L = \frac{1}{2}(l^2\dot\varphi^2 + \dot l^2 + l^2 + 1) + l^2\dot\varphi - lc\dot\varphi - \dot l s - lc + \frac{\rho^3}{r}$$

Here $\rho^3 = fM/\theta^2$ is the cube of the radius of the Kepler orbit, corresponding to the angular velocity $\theta'$, the dot above a symbol denotes differentiation with respect to $\theta$ and we have introduced the dimensionless quantities

$$l = \frac{l_*}{R}, \quad r = \frac{r_*}{R}, \quad \rho = \frac{\rho_*}{R}$$

When $i = 0$ the system allows of a Jacobi integral which, apart from an additive constant, has the form

$$H = \frac{1}{2}l^2\dot\varphi^2 + W, \quad W = lc - \frac{\rho^3}{r}$$

Here $W$ is the augmented potential energy. The equations of motion in this case are completely integrable.

To determine the reactions in the rod we will write Lagrange's equation corresponding to the variable $l$,

$$\ddot l - l\dot\varphi^2 - l - 2l\dot\varphi + c + \frac{\rho^3}{r^3}(l - c) = Q_l \tag{1.3}$$

whence, when $l$ = const, we obtain the tension in the rod

$$N = l\dot\varphi^2 + l + 2l\dot\varphi - c - \frac{\rho^3}{r^3}(l - c)$$

When an inextensible tether is used instead of a rod, the condition that the tension should be positive $N \geqslant 0$ must be satisfied.

In the $(\varphi, \dot\varphi)$ plane the curve of zero reactions intersects the vertical axis $\varphi = 0$ at the points

$$\dot\varphi = -1 \pm \left[l^{-1}\left(1 - \frac{\rho^3}{(1-l)^2}\right)\right]^{1/2}$$

the vertical axis $\varphi = \pi/2$ at the points

$$\dot{\varphi} = -1 \pm \left( \frac{\rho}{\sqrt{1+l^2}} \right)^{3/2}$$

and the axis $\varphi = \pi$ at the points

$$\dot{\varphi} = -1 \pm \left[ l^{-1} \left( 1 - \frac{\rho^3}{(1+l)^2} \right) \right]^{1/2}$$

The latter two points merge if

$$\frac{\rho^3}{(1+l)^2} = 1 \tag{1.4}$$

and disappear if

$$\frac{\rho^3}{(1+l)^2} < 1$$

When condition (1.4) is satisfied two branches of the curve have a common point on the vertical $\varphi = \pi$. The curve $N = 0$ intersects the horizontal axis $\dot{\varphi} = 0$ at points given by the equation

$$(l-c)\left(1 - \frac{\rho^3}{r^3}\right) = 0$$

## 2. STEADY MOTIONS

Using Lagrange's theorem the equilibria of a pendulum with respect to the uniformly rotating system of coordinates can be obtained as the critical points of the changed potential energy

$$\frac{dW}{d\varphi} = ls\left(\frac{\rho^3}{r^3} - 1\right) = 0 \tag{2.1}$$

These equations always allow of two solutions

$$1)\ \varphi = 0,\ 2)\ \varphi = \pi$$

Positions of the pendulum along the descending and ascending verticals respectively, passing through the point $P$, correspond to these.

When $1 - l < \rho < 1 + l$ two solutions with $s \neq 0$ exist (solutions of the form 3). For these solutions the point $Q$ is situated at some point of intersection of a circle with centre $O$ and radius $\rho$ and a circle with centre $P$ and radius $l$. On these solutions $N = 0$.

Note that the form of the equilibria depends considerably on the position of the load with respect to the circle of radius $\rho$ (above or below).

We will estimate the orders of the quantities occurring in this problem. From the expression for a cube of the radius $\rho$ we have

$$V^2 = fM/\rho$$

where $V$ is the linear orbital velocity. By varying the ratio and eliminating the constant $fM$ from it we have

$$\delta\rho/\rho = -2\delta V/V$$

Here $\rho = 6 \times 10^6$ m, $V = 9 \times 10^3$ m/s and $\delta V = 0.1$ m/s. Then $\delta\rho \cong 133$ m. Hence, an increment in the linear velocity of 0.1 m/s gives a change in $\rho$ by an amount comparable with the length of the pendulum.

## 3. THE STABILITY OF RELATIVE EQUILIBRIA

We will use Lagrange's theorem to investigate the stability of the relative equilibria. The second derivative of the changed potential has the form

$$W'' = \left(\frac{\rho^3}{r^3} - 1\right) lc - \frac{3\rho^3}{2r^5} l^2 s^2$$

In solutions 1, 2 and 3 this derivative has the values

$$W_1'' = \frac{\rho^3 - (1-l)^3}{(1-l)^3} l \geqslant 0 \quad \text{for } \rho \geqslant 1 - l \text{ (and } l < 1)$$

$$W_2'' = -\frac{\rho^3 - (1+l)^3}{(1+l)^3} l \geqslant 0 \quad \text{for } \rho \leqslant 1 + l$$

$$W_3'' = -\frac{3\rho^3}{2r^5} l^2 s^2 < 0 \quad \text{for } 1 - l < \rho < 1 + l$$

The bifurcation diagram is shown in Fig. 2. The continuous lines correspond to stable solutions and the dashed lines correspond to unstable solutions.

If $\omega_{1\pm l}$ is the Kepler angular velocity, corresponding to the angular Kepler orbit of radius $1 \pm l$, then when $l < 1$ the stability conditions can be represented in the form

$$1)\ \omega_{1-l}^2 - 1 < 0, \quad 2)\ \omega_{1+l}^2 - 1 > 0 \qquad (3.1)$$

Solutions of the form 3 are always unstable for $1 - l < \rho < 1 + l$.

## 4. DIAGRAM OF THE MOTIONS OF A PENDULUM IN THE PHASE PLANE

The phase portrait of the system for different values of the parameters $l$ and $\rho$ is shown in Fig. 3. When $\rho = 1$ we have a pattern which is qualitatively identical with that obtained previously in [6]. This and all the subsequent diagrams can be extended to the left and to the right by symmetrical reflection in the vertical straight lines $\varphi = \pi k$ ($k = 0, \pm 1, \ldots$). The first diagram is symmetrical about the axis $\varphi = \pi/2$. The points on the axis $\dot\varphi = 0$ with coordinates $\varphi = 0$ and $\varphi = \pi$ correspond to stable relative equilibria, whereas the point $\varphi \equiv \pi/2$ corresponds to unstable relative equilibrium. The oval passing through this point corresponds to a zero value of the tension in the rod; inside the oval the rod is compressed ($N < 0$), i.e. these equilibria cannot be obtained using a tether.

We will now consider the case when $\rho = 1 \pm \varepsilon$, $0 < \varepsilon \ll 1$. The pattern in this case becomes asymmetrical. The two stable equilibria remain in place, but the unstable equilibrium is shifted along the abscissa axis. The oval, while still passing through the saddle point and close to the point $\varphi = \pi/2$ on the abscissa axis, loses its symmetry (Fig. 3 with $\rho = 1.001$ and $l = 0.002$). If we now reduce the value of $l$, then for a certain value of $l$ the regions $N < 0$, situated in the regions $[0, \pi] \times \dot\varphi$ and $[\pi, 2\pi] \times \dot\varphi$ become closed and occupy the interior of a "figure of eight" (Fig. 3 with $\rho = 1.001$ and $l = 0.0015$).

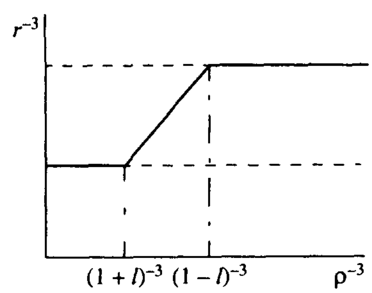


Fig. 2

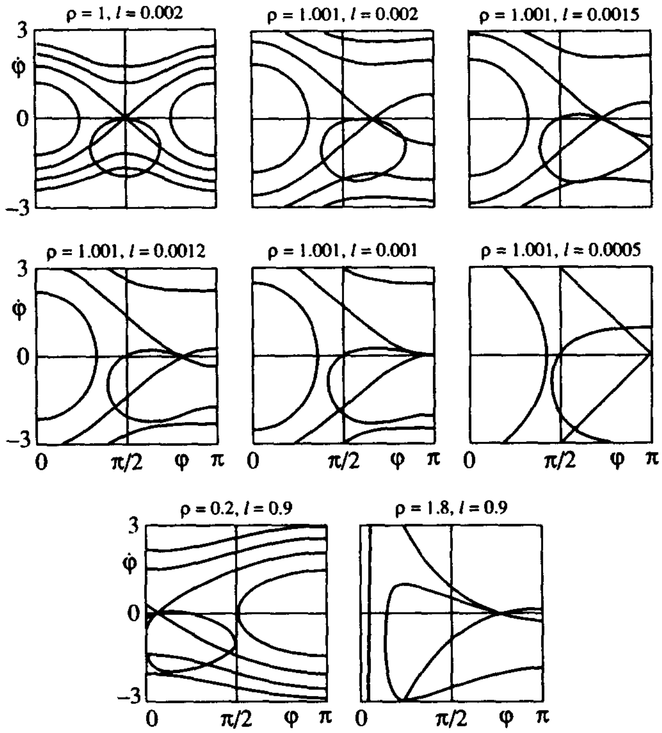


Fig. 3

When $l$ is reduced further, the "figure of eight" is converted into a symmetrical curve (with axis of symmetry $\varphi = \pi$), diffeomorphic to a circle, but still passing through the saddle point (Fig. 3 for $\rho = 1.001$ and $l = 0.0012$). Further, the stable relative equilibrium with $\varphi = \pi$ becomes unstable, being on the boundary of the compression region of the rod (Fig. 3 with $\rho = 1.001$ and $l = 0.001$). Finally, when $l$ is reduced further the unstable relative equilibrium turns out to be inside the region $N < 0$ (Fig. 3 for $\rho = 1.001$ and $l = 0.0005$). Calculations show that for small values of $l$ and $\varepsilon$, corresponding to actual satellite systems, the patterns constructed for $\rho = 1 + \varepsilon$ and $1 - \varepsilon$ with graphical accuracy, can be obtained from one another by symmetrical reflection in the straight line $\varphi = \pi/2$. If this is not the case, this symmetry disappears (Fig. 3 with $\rho = 0.2$ and $1.8$ and $l = 0.9$).

## 5. INTERPRETATION OF THE RESULTS

Suppose the centre of mass of the orbital station moves along a circular Kepler orbit (the dimensionless orbital angular velocity is equal to unity) and the orientation of the station is unchanged with respect to an orbital system of coordinates. Suppose a crane consisting of a circular directrix with centre at $O$ of radius $R$ is set up on an orbital station, and from the carriage $P$ it is possible to slide along this directrix and from a load $Q$. This directrix can be regarded locally as rectilinear. The orbital angular velocity of the carriage $\dot{\theta} = 1 + \lambda$ consists of the translational angular velocity, which is equal to unity, and the relative angular velocity $\lambda$.

This device enables the load to be displaced in the orbital plane, and when motion occurs with constant relative velocity the load can be situated either below or above the suspension point, both positions being stable. By virtue of the first condition of (3.1) the lower position is stable if

$$\lambda < \omega_{R-l} - 1$$

This condition imposes a limitation on the velocity of the carriage motion along the direction of orbital motion.

By virtue of the second condition of (3.1) the upper position is stable if

$$\lambda > \omega_{R+l} - 1$$

This condition imposes a limitation on the velocity of the carriage motion in a direction opposite to the direction of orbital motion.

This research was supported by the Hochschule Jubilaümstiftung der Stadt Wien, the Russian Foundation for Basic Research (99-01-00785, 00-15-96150 and 01-01-02001), the Federal Special-Purpose "Integration" Programme and the Belgian Ministry for Foreign Affairs.

## REFERENCES

1. McQUADE, F. and McINNES, C. R., Co-operative control for on-orbit assembly of large space structures using potential field methods. Departmental Report No. 9818. Department of Aerospace Engineering Glasgow: University of Glasgow, Glasgow, 1998.
2. CHERNOUS'KO, F. L., The optimum displacement of a pendulum. *Prikl. Mat. Mekh.*, 1975, **39**, 5, 806–816.
3. SYNGE, J. L., On the behaviour, according to Newtonian theory, of a plumb line or pendulum attached to an artificial satellite. *Proc. Roy. Irish Acad.*, 1959, **20**, Ser. A., 1–6.
4. BLITZER, L., Equilibrium and stability of a pendulum in an orbiting spaceship. *A. J. Phys.*, 1979, **47**, 241–246.
5. BUROV, A. A. and TROGER, H., The relative equilibria of an orbital pendulum suspended on a tether. *Prikl. Mat. Mekh.*, 2000, **64**, 5, 755–760.
6. BELETSKY, V. V. and LEVIN, E. M. *Dynamics of Space Tether Systems. Advances in the Astronautical Sciences*, Vol. 83. American Astronautical Society, San Diego, CA, 1993.

*Translated by* R.C.G.